Traditional microwave stability analysis proves to be insufficient
(memo)

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Abstract— **From a practical point of view it is very important to have an electronic circuit which is unconditional stable, ie. it will not produce unwanted additional frequency components by itself. This seems trivial but to predict this unwanted behavior is not trivial at all. In microwave history there is one well established method to determine unconditional stability. This is based on the work of Stern and others. In this memo, this will be proved to be incorrect and as a consequence the reader is encouraged to look into more fundamental methods in determining stability. The underlying method to accomplish this is the principle of the argument theorem of complex theory. Also Nyquist used this same theorem in his stability analysis. The author hopes to contribute to the awareness of the shortcoming of traditional Stern based stability analysis via a counter example, without providing a mathematical prove to show the actual flaw.**

I. Introduction

It is common within the RF community to use stability factors to determine the stability of a linear time-invariant two-port, especially in the microwave field. The most used equation, as function of the scattering parameters, is

$$k = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta S|^2}{2\,|S_{12}S_{21}|}, \tag{1}$$

where for unconditional stability the following must apply:

$$k > 1 \text{ and } |\Delta S| < 1, \tag{2}$$

where $\Delta S = S_{11}S_{22} - S_{12}S_{21}$. Often this type of stability calculation is simply referred to as "*k-factor*". A disadvantage of this method is that it uses two conditions and as a result, a more sophisticated stability factor has been defined [1], where stability is ensured by a single condition, or

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S\, S_{11}^*| + |S_{12}\, S_{21}|} > 1. \tag{3}$$

In this memo, an example will be given where the generally accepted method(s) will fail. A more detailed discussion of this phenomena is given in [2], which upon this memo is based. The most important cause of failure is that analysis is done on a two-port which is a reduced version from larger N-node networks. Unfortunately, this condition is true in practise for most cases.

This memo is primarily intended to make designers aware of the shortcomings of the well established methods for determining stability.

II. A Simple Ring Oscillator as Example of Failure

Fig. 1 shows the schematic diagram of the ring oscillator under test.

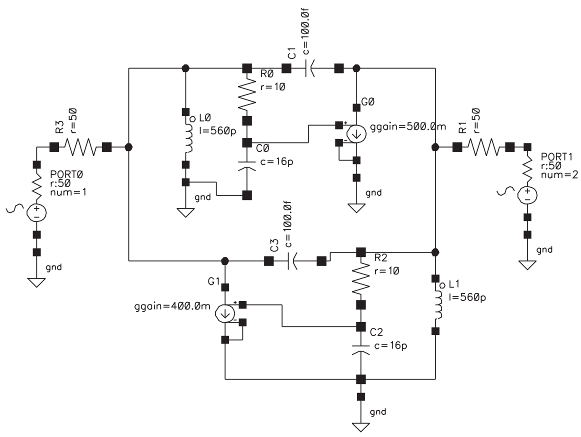

Fig. 1. Schematic diagram representation in SpectreRF.

Fig. 2 shows the Rollet stability factor *k*.

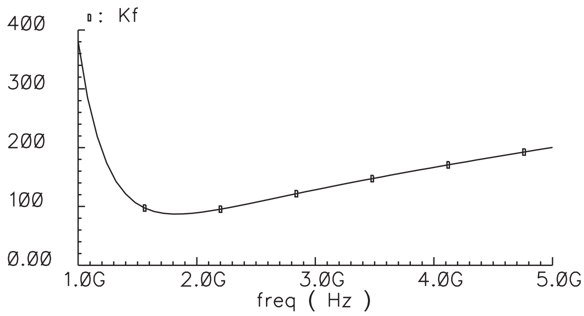

Fig. 2. Rollet stability factor, stable if $k > 1$.

For stability there is a second condition to be fulfilled. The determinant of the scattering parameters should be smaller than one. Fig. 3 shows this.

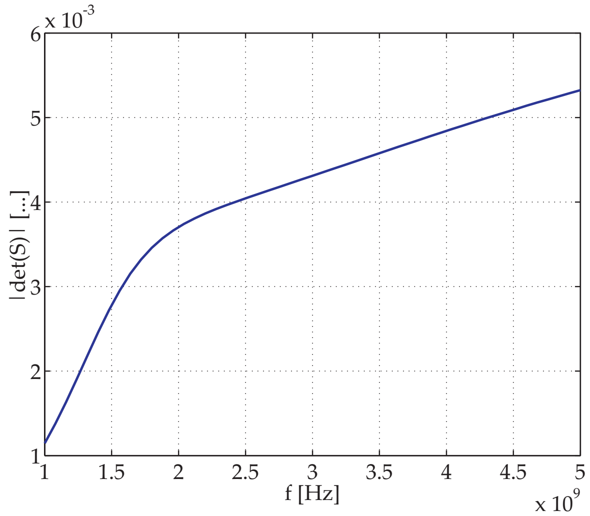

Fig. 3. Determinant of the S-parameters, for stability this should be less than 1.

Based on this, one must conclude that the two-port under test is unconditional stable. To investigate the stability with a given source and load impedance, the stability factor of Stern can be used. Fig. 4 shows this factor.

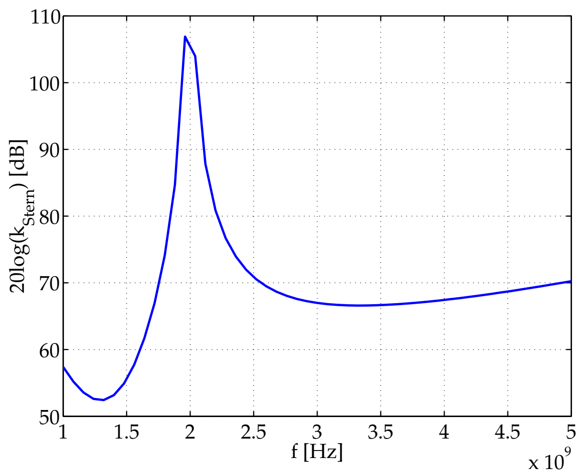

Fig. 4. Stern stability factor, stability if larger than 0 dB.

It is clear that the ring oscillator should be stable. Please note that for readability reasons a logarithmic y-axis has been chosen. There is a tendency of the graph around 1.3 GHz but this is no strong indication of instability. Also the alternative stability factor shows an unconditional stable circuit, see Fig. 5.

Finally, a third method is investigated to determine

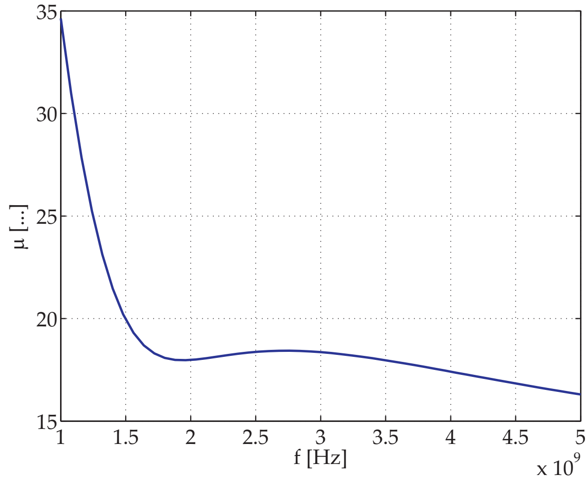

Fig. 5. Alternative stability factor, stable if $\mu > 1$.

the stability. This is done via the eigenvalues of

$$S^{\dagger} \cdot S, \tag{4}$$

where $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ and where the dagger denotes Hermitian conjugate or conjugate transpose. This third method can be explained as follows [3]. If the amplitude of the two ingoing signals is denoted by $\vec{\alpha} = (\alpha_1, \alpha_2)$, and the outgoing signals by $\vec{\beta} = (\beta_1, \beta_2)$, then

$$\vec{\beta} = S \cdot \vec{\alpha} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \tag{5}$$

Energy conservation requires $|\vec{\beta}| \leq |\vec{\alpha}|$. Using the singular value decomposition for the matrix S, we find: (with U_1 and U_2 unitary matrices, and $\lambda_{\pm}$ positive real numbers)

$$\vec{\beta} = S \cdot \vec{\alpha} = U_1 \cdot \begin{bmatrix} \lambda_+ & \\ & \lambda_- \end{bmatrix} \cdot U_2 \cdot \vec{\alpha}. \tag{6}$$

Energy conservation now comes down to $\lambda_+, \lambda_- \leq 1$. One can immediately see that the squares of the singular values are the eigenvalues of the matrix $S^{\dagger}S$, and for the latter we can find an expression based on the characteristic polynomial:

$$\lambda_{\pm}^2 = e_{\pm} = \frac{\mathrm{Tr}\,\left(S^{\dagger}S\right) \pm \sqrt{\mathrm{Tr}\,\left(S^{\dagger}S\right)^2 - 4\det\,\left(S^{\dagger}S\right)}}{2} \tag{7}$$

Where Tr $(.)$ denotes the trace or sum of the diagonal elements of the matrix. To obtain a stability factor based on the singular value decomposition, we should look at the largest of the eigenvalues only. Since both of them must be real ($S^{\dagger}S$ is hermitic) we know the square root in the previous expression is real, so taking the positive sign will give the value of e_+.

We can also write out the terms, by using

$$S^\dagger S = \begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix} \cdot \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}^* S_{12} + S_{21}^* S_{22} \\ S_{11} S_{12}^* + S_{21} S_{22}^* & |S_{12}|^2 + |S_{22}|^2 \end{bmatrix}, \quad (8)$$

which leads to

$$\mathrm{Tr}\left(S^\dagger S\right) = |S_{11}|^2 + |S_{21}|^2 + |S_{12}|^2 + |S_{22}|^2$$
$$\det\left(S^\dagger S\right) = \left(|S_{11}|^2 + |S_{21}|^2\right)\left(|S_{12}|^2 + |S_{22}|^2\right) - |S_{11}^* S_{12} + S_{21}^* S_{22}|^2 \quad (9)$$

and finally gives us:

$$e_+ = |S_{11}|^2 + |S_{21}|^2 + |S_{12}|^2 + |S_{22}|^2 + \sqrt{\left(|S_{11}|^2 + |S_{21}|^2 - |S_{12}|^2 - |S_{22}|^2\right)^2 + 4\,|S_{11}^* S_{12} + S_{21}^* S_{22}|^2} \quad (10)$$

with the stability criterion $e_+ \leq 1$. This outcome is verified in Mathematica 5.0 and is as follows.

$$Eigenvalue_{1,2} = \tfrac{1}{2}\left(|S_{11}|^2 + |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2 \pm \sqrt{\left(|S_{11}|^2 + |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2\right)^2 - 4\,|\Delta S|^2}\right). \quad (11)$$

If at least one of the eigenvalues is larger than one, it means that there is at least one real pole in the right half plane. This clearly indicates instability. It also reveals the shortcoming of this method since it concentrates on the real axis only. Complex poles in the right half plane are not discovered by this method. Fig. 6 shows the $\sqrt{e_\pm}$.

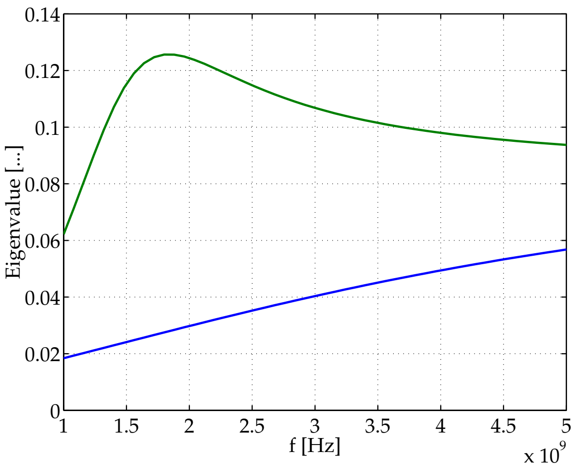

Fig. 6. $\sqrt{e_\pm}$ of the S-parameters, for stability both should be less than 1.

So far, there is no indication that the two-port under test is instable. Based on the analysis one must conclude that the circuit is unconditionally stable. However, this is not true and this will be demonstrated in the next section.

III. Proper determination of stability

There are two rigorous methods known to the author that can be used to determine a system's stability. The first is a straightforward one, ie. a transient analysis. Here the assumption is made that the transient simulator is capable enough to handle instable circuits. Sometimes initial conditions are required to show instability effects and in other cases a proper integration method should be selected. The second method is to find poles in the right hand plane. This is also the method used in [2] which paper upon this memo is based. The underlaying mathematical theorem is the same as used by Nyquist [4] for his stability analysis. A full explanation on how to determine the poles of an N-node circuit is far beyond the scope of this memo. Please refer to [2] for a detailed discussion on this topic. However, in the SpectreRF simulator there is a possibility to display the poles of a two-port under test. In this particular case this analysis already shows clearly that there is indeed a stability problem, since there are two complex conjugated poles to be found in the right half plane. Fig. 7 shows the location of the poles found by this simulator.

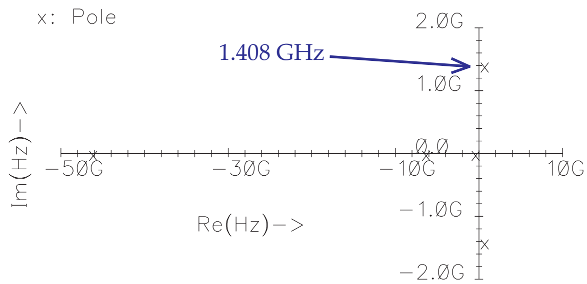

Fig. 7. The location of the poles found by SpectreRF.

This result resembles the outcome of the analysis conducted in [2], where a frequency of oscillation of 1.4075 GHz was determined.

Fig. 8 shows the transient behavior if the circuit under test has been excited with a very short voltage pulse in series with one of the inductors. This is done to initiate the instability since there is no (omnipresent) noise available during a transient analysis.

As can be seen from the transient analysis picture, the half period time is approximately 357.592 $psec$. This corresponds to a natural frequency of 1.4 GHz. This is pretty close to the value found in [2].

IV. Conclusion

It has been shown that the traditional method of determining stability based upon the foundations described by Stern can fail for the case where the circuit under test is a reduced version of a larger N-node network. It

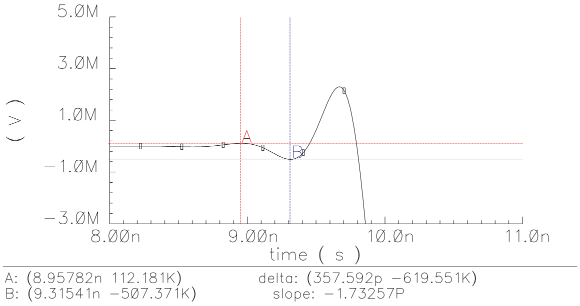

Fig. 8. Transient analysis with two (crosshair) markers to determine the natural frequency.

has also be shown that a proper analysis can be done through a well established pole location analysis where it is important for stability that there are no poles in the right hand plane. The author hopes to contribute to the awareness of the shortcoming of traditional Stern based stability analysis via a counter example, without providing a mathematical prove to show the actual flaw.

Acknowledgment

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